Exercise 8

(a) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

Solution

Part (a)

$$(x, y) + (u, v) = (x, y)$$

Adding the two complex numbers on the left side, we get

$$(x + u, y + v) = (x, y).$$

In order for equality to hold, the real and imaginary parts of both sides must be equal.

$$\left.\begin{aligned} x + u &= x \\ y + v &= y \end{aligned}\right\}$$

This is a system of two equations for two unknowns, u and v. Subtract both sides of the first equation by x, and subtract both sides of the second equation by y.

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\}$$

Therefore, $(0, 0) = 0 + 0i$ is the unique additive identity for complex numbers.

Part (b)

$$(x, y)(u, v) = (x, y)$$

Multiplying the two complex numbers on the left side, we get

$$(xu - yv, uy + xv) = (x, y).$$

In order for equality to hold, the real and imaginary parts of both sides must be equal.

$$\left.\begin{aligned} xu - yv &= x \\ uy + xv &= y \end{aligned}\right\}$$

This is a system of two equations for two unknowns, u and v. Solve the first equation for y,

$$xu - x = yv$$

$$x(u - 1) = yv$$

$$\frac{x(u-1)}{v} = y, \tag{1}$$

and substitute it into the second equation.

$$uy + xv = y$$

$$uy - y = -xv$$

$$y(u-1) = -xv$$

$$\left[\frac{x(u-1)}{v}\right](u-1) = -xv$$

$$x(u-1)^2 = -xv^2$$

$$x(u-1)^2 + xv^2 = 0$$

$$x[(u-1)^2 + v^2] = 0$$

By the zero product property of real numbers,

$$x = 0 \quad \text{or} \quad (u-1)^2 + v^2 = 0.$$

Plugging $x = 0$ into equation (1) gives $y = 0$, so in the special case that $(x, y) = (0, 0)$, u and v are arbitrary. However, for general x and y, only $u = 1$ and $v = 0$ satisfy the equation on the right. Therefore, $(1, 0) = 1 + 0i$ is the unique multiplicative identity for complex numbers.